Journal of Sound and Vibration (1997) **207**(3), 383–392

NON-LINEAR VIBRATIONS OF HARDENING SYSTEMS: CHAOTIC DYNAMICS AND UNPREDICTABLE JUMPS TO AND FROM RESONANCE

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(*Received* 19 *September* 1996, *and in final form* 8 *May* 1997)

The non-linear resonance behaviour of two typical hardening systems subjected to various forms of excitation is examined. For the system with a linear stiffness component it is shown that for small forcing levels the system behaves like a linear system with resonance occurring when the forcing frequency is approximately equal to the linearized natural frequency. As the forcing amplitude is increased the steady state response peaks towards higher frequencies leading to the well known *jump phenomenon*. Often such jumps to (and from) resonance are a purely deterministic event in which the system settles on to the solution lying on the (non-)resonant branch of the response curve. At higher, but still moderate forcing values, it is shown that such jumps can be *indeterminate* in the sense that one cannot predict whether the system re-stabilizes or not; indeed, their outcome may go to another coexisting solution at the bifurcation. Examples of hardening systems exhibiting unpredictable jumps *to* and *from* resonance are presented.

1. INTRODUCTION

In non-linear vibrations, hardening systems can loosely be classed as systems the stiffness of which increases with load. Although hardening systems are similar to linear systems in that they are characterized by having one statical equilibrium state, such that autonomous motions are confined to a single potential well (see Figure 1), their associated resonance behaviour differs quite substantially [1–5]. It has been shown that in situations in which there is more than one remote coexisting attractor at the local instability, jumps may be indeterminate in the sense that one cannot predict to which solution the system will settle upon, in which the outcome may be probabilistic [6]. This followed earlier work on the response of softening systems [7, 8] who examined the motions of a particle that has the ability to escape from a potential well in which there inherently exists a remote coexisting solution at a local bifurcation [7–9]. The basin and manifold organization generating indeterminate bifurcations have been discussed by Soliman and Thompson [8]. It was shown that when a highly intertwined basin structure accumulating in the vicinity of the saddle-node due to the inherent uncertainties in the specification of the initial conditions or parameter values, long-term predictability will be lost and hence the jump will be indeterminate. Indeed irrelevant of the type of non-linearity, whether it be softening or hardening, it is the structure of the attractor–basin phase portrait that dictates whether or not there is an indeterminate bifurcation. In reference [10], bifurcations of equilibria for hardening systems were examined; here bifurcations of cycles which exhibit the more

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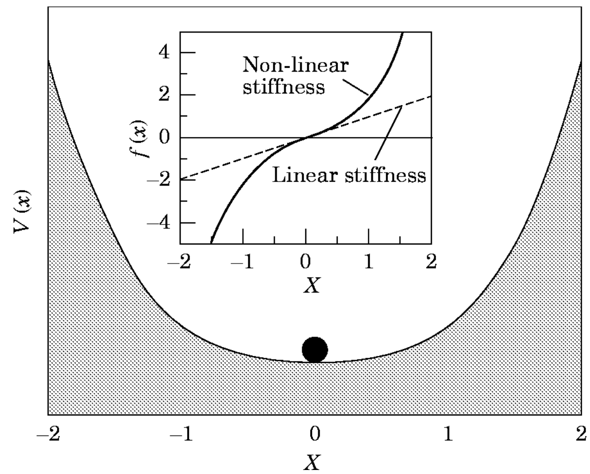

Figure 1. Motions in a hardening system.

typical resonance behaviour are discussed. In order to illustrate these ideas we consider two systems which are modelled by a particle in a potential well governed by the potential energies $V(x) = x^2/2 + x^4$ and $V(x) = x^4/4$, the corresponding equations of motion of which are

$$\ddot{x} + \beta\dot{x} + (1 + F\sin\omega t)\,(x + x^3) = 0{\cdot}1F\sin(\omega t) \qquad \text{(model A)},$$

$$\ddot{x} + \beta\dot{x} + x^3 = F\cos(\omega t) \qquad \text{(model B)},$$

which, as indicated, are referred to as model A and model B respectively. Here, x is the dependant variable and a dot denotes differentiation with respect to time t. The magnitude of the damping, β, is fixed at $\beta = 0{\cdot}5$. For model A, the oscillator is driven by an excitation of amplitude F and frequency ω. The focus of the studies here is on the region near the fundamental frequency regime ($\omega \approx 1$). The studies are concentrated on model A unless otherwise explicitly stated. Note that model A is both parametrically and externally excited, unlike the parametrically excited system examined by Soliman and Thompson [10]. Here bifurcations are generic since in the parametrically excited system, bifurcations are of course structurally unstable, since a typical perturbation of the dynamical system will destroy the equilibrium solution [11].

The structure of this paper is as follows. Section 2 is a discussion of the implications of indeterminate jump phenomenon for engineers and applied scientists. Section 3 is an outline of the main steady state bifurcations, in (F, ω) control space, in the frequency regime close to $\omega = 1$ for model A. Section 4 presents non-linear resonance curves of the above system at relatively small forcing levels. *Safe* jumps are illustrated by determining the attractor–basin phase portraits just prior to the jump; it is shown that at the point of instability the saddle-node is located in the interior of the resonant basin well away from the basin boundary separating any remote attractors. Section 5 delineates the phenomena of *indeterminate jumps*; here it is shown that the saddle-node will hit a highly intertwined basin boundary at the point of instability: long-term predictability will be lost and hence the jump will be indeterminate in the sense that the system settles on to any of the attractors the basins of which have accumulated in the vicinity of the saddle-node. Two examples (model A and model B) illustrating unpredictable jumps *to* and *from* resonance are presented. In has to be noted that the addition of parametric excitation in model A tends to have a destabilizing effect on the resonant branch of the response curve and the creation of several large amplitude attractors; here the focus is on indeterminate jumps to resonance. On the other hand for the purely externally excited hardening system (model B) jumps to resonance are found to be deterministic since the only other attractor is the resonant solution lying on the resonant branch of the response curve; attention therefore is focussed on frequencies generating jumps from resonance in which multiple solutions may coexist. In section 6 *unsafe* jumps are discussed, in which there is no possibility of the system re-stabilizing on to the resonant solution. The outcome may or may not be determinate. Conclusions are in section 7.

2. IMPLICATIONS OF INDETERMINATE JUMP PHENOMENON FOR ENGINEERS

As shall be shown, one of the classical features exhibited in non-linear structural and mechanical systems is the jump phenomenon, in which under the variation of a control parameter, such as the frequency, there is jump to a disconnected attractor, resulting not only in a qualitative, but often a substantial quantitative change in the response. It will be shown that in situations in which there is more than one remote coexisting attractor at the local instability, jumps may be indeterminate in the sense that one cannot predict to which solution the system will settle upon. This has important implications for engineers and applied scientists involved with physical experiments, simulation of numerical models or analytical studies. For example the experimentalist when dealing with real physical systems, where it is difficult to control the imperfections, such as the material inhomogeneities, the initial conditions or parameter transitions with an indeterminate outcome imply that repetition under 'almost identical' conditions may result in qualitatively different types of response. In situations in which parameters are slowly varied, a loss of repeatability may be interpreted in the context of the inherent indeterminacy properties of the system, rather than be attributed to experimental error [12]. This may equally be applied for the engineer carrying out numerical experiments. Furthermore, analytical techniques may not be able to pick up such indeterminate phenomena since they become increasingly inaccurate as the strength of the non-linearity is increased in which chaotic oscillations, or coexisting basins of attraction with an intricate structure, may occur. In such cases the analyst has to resort to numerical techniques. It has to be pointed out that recently there have been laboratory experiments [13] to confirm many features of the unpredictability of the jump phenomenon.

3. BIFURCATION BOUNDARIES

In Figure 2 are shown the main steady state stability boundaries, in (F, ω) control space, in the frequency regime close to $\omega = 1$. Point S is a cusp; for this hardening system, frequencies below ω^S or forcing values below F^s, no jump occurs. Above S, there is the creation of two fold lines. A represents a saddle-node bifurcation, which under increasing F, results in a *jump to resonance*. Line B corresponds to a saddle-node fold which results in the creation of a finite amplitude limit cycle of order $n = 1$. Line C is the first period-doubling flip bifurcation, at which this resonant harmonic attractor period-doubles to a stable subharmonic of order $n = 2$. There is an infinite sequence of these flip bifurcations leading to a chaotic attractor which finally loses its stability at a crisis at E. Point Q corresponds to where line E intersects line A. For jumps to resonance occurring from a saddle-node bifurcation (from A) above point Q, there are no available fundamental attractors to jump to, and hence unsafe jump occurs. The steady state stability boundary, beyond which there are no fundamental attractors within the well, is indicated by the dot-screen. Several analytical techniques may be used to predict

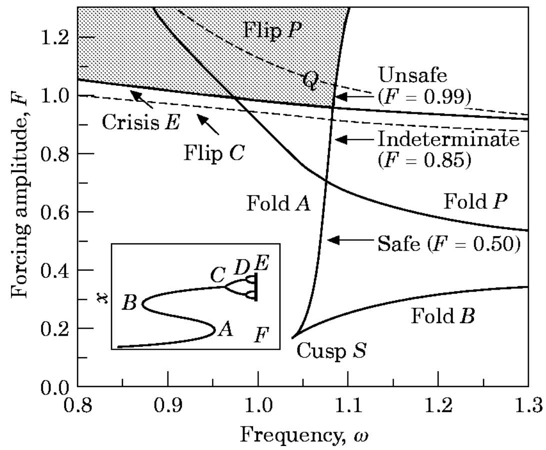

Figure 2. Bifurcation diagram (model A). The bifurcation diagram in the (F, ω) control space at $\beta = 0{\cdot}05$. Line A indicates the loss of stability at a saddle-node bifurcation the fundamental solution originating from $F = x = y = 0$. Line B is the fold creation of the resonant limit cycle, C is the first flip bifurcation of this attractor; D is the second period-doubling bifurcation approximating line E where this attractor loses its stability at a crisis. Lines P correspond to bifurcation values of a coexisting period-1 attractor.

instabilities A and E and hence determine this region. Most notably a technique has been developed by Virgin [14], who examined equations of this type and obtained good comparison with results given by numerical techniques. It was shown in that paper that fold line A was determined from a first order instability criterion; period-doubling line C was obtained by using Floquet theory to predict an instability of a period-T oscillation, giving rise to the period-$2T$ solution (which approximates line E).

It has to be acknowledged that what have been determined here are the bifurcation lines for the fundamental solution (that originating from $F = x = y = 0$); in typical non-linear systems multiple solutions may exist [15]. Hence corresponding bifurcation lines of a typical coexisting solution, a period-1 attractor, have been superimposed.

4. SAFE DETERMINATE JUMPS: RELATIVELY SMALL FORCING AMPLITUDES

In Figure 3 are shown, for relatively small F, response curves in the frequency–amplitude plane illustrating the hardening characteristics of the system. At the lowest value of forcing

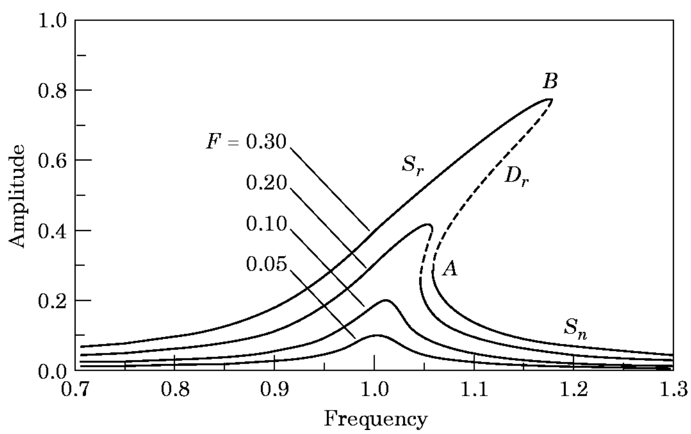

Figure 3. Non-linear resonance in hardening systems (model A).

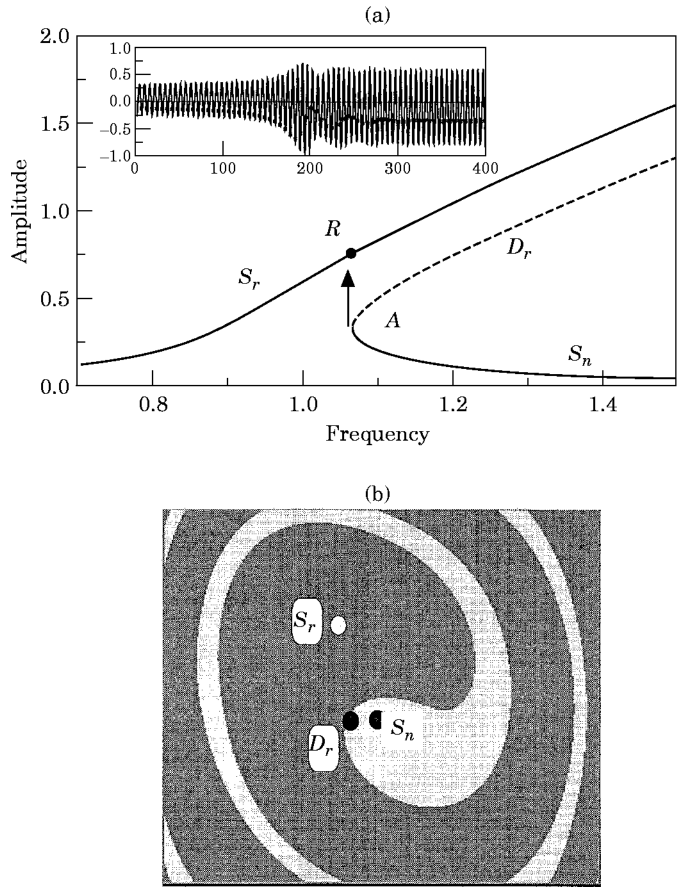

Figure 4. Safe determinate jumps (model A). (a) The steady state resonance response curve at a relatively low forcing level ($F = 0{\cdot}50$), illustrating how the amplitude of the response, X, varies with frequency, Ω. Here the jump from point A always re-stabilizes on to the large amplitude limit cycle, R, that lies on the resonant branch of the response curve. The inset shows a typical time history after the fold; here the transient response always leads to the large amplitude limit cycle. (b) The attractor–basin phase portrait just prior to a safe saddle-node bifurcation in the window $-1{\cdot}5 < x < 1{\cdot}5$, $-1{\cdot}5 < y < 1{\cdot}5$. Here $\omega = 1{\cdot}0775$, $G = 0{\cdot}50$. S_r represent the Poincaré, points of the stable $n = 1$ harmonic attractor; D_r represents the unstable saddle solution and S_r represents the non-resonant solution. Grey shading represents the basin of attraction of the resonant solution; the basin of the non-resonant solution is in white.

amplitude considered ($F = 0{\cdot}05$), resonance occurs when the forcing frequency is approximately equal to the linearized natural frequency. As F is increased the steady state response peaks towards higher frequencies, typical of hardening systems. At $F = 0{\cdot}3$ the classical *jump phenomenon* takes place; upon slowly increasing ω from relatively small values there is a *jump from resonance* at fold point B; conversely under increasing ω there is a *jump to resonance*, at point A which results in a substantial quantitative change in the amplitude of the response. At larger values of F it may be noted, from the bifurcation diagram of Figure 2, that for $0{\cdot}35 < F < 0{\cdot}85$ the harmonic $n = 1$ solution lying on the upper branch of the response curve remains stable for all frequency values considered here. This can be seen in Figure 4(a) showing the response diagram at $F = 0{\cdot}50$. Under slowly increasing ω the harmonic cycle grows in amplitude. Conversely, upon decreasing ω from a relatively high value, one observe that the system loses its stability at a saddle-node bifurcation, at point A. Here, as in the previous case, small perturbations from point A will result in a jump to resonance which *always* re-stabilizes on to the resonant oscillation

R, lying on the resonant branch of the response curve. A typical time history is shown in the inset; here the system has been set on the saddle-node and then given a small increment $\Delta\omega$. We see that the system initially remains near the now unstable saddle-node solution, and finally settles on to the steady state large amplitude limit cycle. This is due to the fact that no major global connections between the unstable manifold of the resonant saddle, D_r, and stable manifold have occurred [10]. This is clearly delineated in Figure 4(b), which shows the basin organization, in the Poincaré section, just prior to the jump to resonance at A. Here it can clearly be seen that when the saddle-node bifurcation occurs, in which the non-resonant solution, S_n, coalesces with the unstable solution, represented by the mapping points D_r, the system will re-stabilize on to the resonant attractor S_r as it is located in the interior of the resonant basin. Thus one has a totally determinate jump to S_r which is preserved under conditions of small external noise or finite perturbation of the initial conditions.

5. INDETERMINATE JUMPS IN HARDENING SYSTEMS

In Figure 5(a), at $F = 0{\cdot}85$, is shown a response diagram in which the jump to resonance is indeterminate. Under decreasing frequency the non-resonant solution loses its stability at the saddle-node, A. At the parameter value of this bifurcation there also coexist other bounded "remote" attractors. Remote attractors in the figure refer to other coexisting attractors not orginating from the fundamental path $F = x = \dot{x} = 0$, notably a period-1 harmonic cycle and a period-3 attractor, as well as other attractors. Examining the basin structure just before the saddle-node at A, one sees that the saddle-node will hit a highly intertwined boundary at the point of instability (Figure 5(b)). From previous studies it has been shown [10] that the unstable manifold of the resonant saddle D_r is heteroclinically tangled with the stable manifolds (basin boundary) of the unstable period-1; this results in an accumulation of the S_r basin, coexisting period-1 and period-3 basins of attraction on to the stable manifold of D_r. If ω were decreased slowly at an infinitesimal rate, the system would find itself at A, sitting on an *infinitely* intertwined boundary [9]. In any real situation, due to the inherent uncertainties in the specification in the parameter values and indeed the value of the perturbation from the saddle-node, long-term predictability will be lost and hence the jump will be indeterminate. Here the system experiences a long transient leading either to the stable resonant limit cycle, S_r, the coexisting period-1 or period-3 attractors [6]. Three possible outcomes under three slightly different perturbations are shown in Figure 5(c). The first time history shows the jump settling on to the fundamental solution, S_r; the second trace, at the same parameter values but from a slightly different frequency values, leads to the coexisting period-1 attractor; the final trace settles on to the period-3 attractor. A more realistic situation has been presented by Soliman [12] where parameters are varied at a finite rate through similar bifurcations

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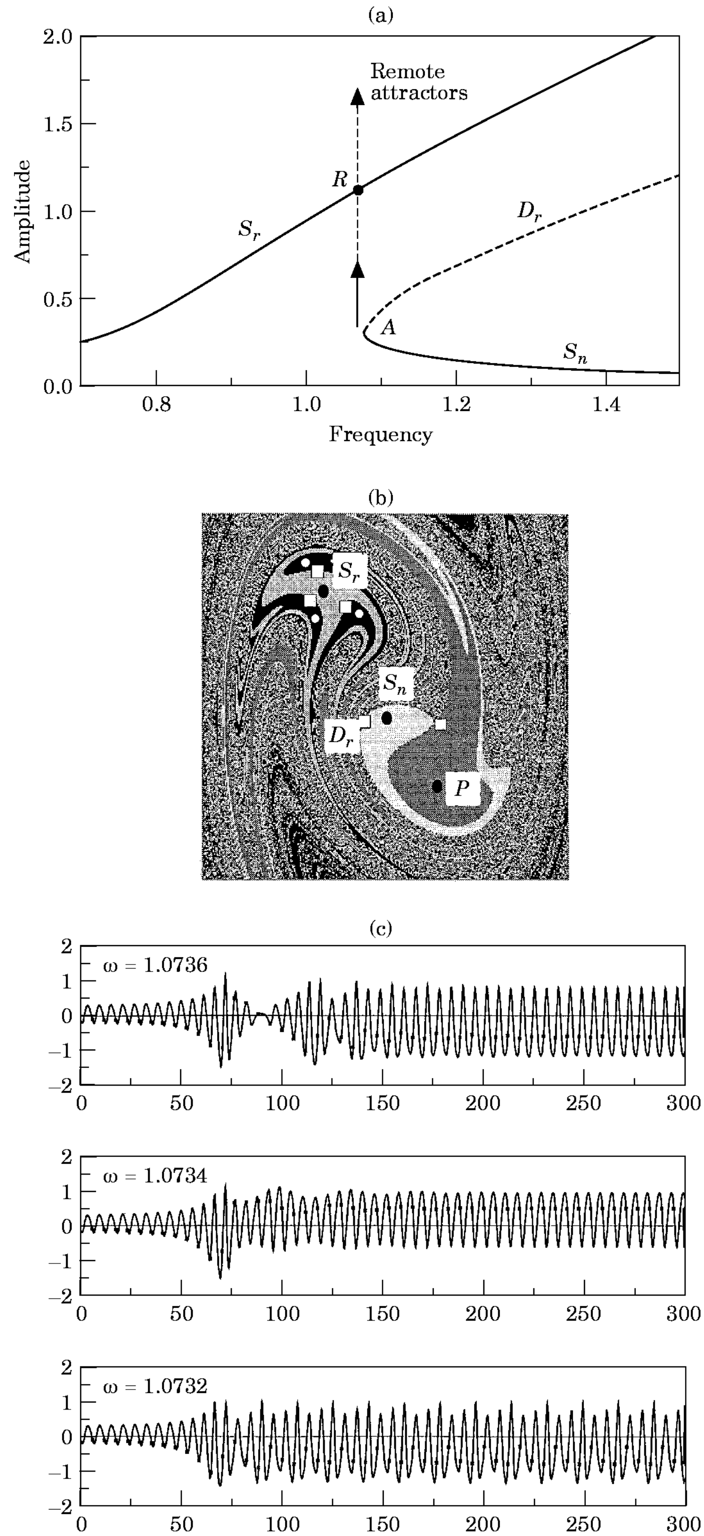
Figure 5. Indeterminate jumps (model A). (a) The steady state resonance response curve at an intermediate forcing level ($F = 0{\cdot}85$). Here the jump from point A may or may not re-stabilize on to the large amplitude limit cycle, R that lies on the resonant branch of the response curve (model A). (b) The upper portrait shows the basin structure just prior to an indeterminate subcritical bifurcation in the window $-1{\cdot}8 < x < 1{\cdot}8$, $-1{\cdot}8 < y < 1{\cdot}8$. Here $\omega = 1{\cdot}09$, $F = 0{\cdot}85$. The circles denote the stable mapping points of the fixed points; the squares the unstable point. The colour coding is as follows: light grey represents the basin of attraction of the resonant solution, S_r; white represents the basin of the non-resonant solution, S_r; grey is for a coexisting period-1 solution, P; black basins of all other remote attractors including a period-3 attractor; (model A). (c) Three possible time histories of $x(t)$ showing the transient response leading to (i) the large amplitude limit cycle, S_r, and (ii) a coexisting period-1 attractor, P, and (iii) a period-3 attractor. In all cases the system is set on the saddle-node and then perturbed by a small increment in frequency (model A).



(a)
Remote
attractors
R
S_r
D_r
A
S_n
Amplitude
Frequency
2.0
1.5
1.0
0.5
0.0
0.8
1.0
1.2
1.4
(b)
S_r
S_n
D_r
P
(c)
ω = 1.0736
ω = 1.0734
ω = 1.0732
0
50
100
150
200
250
300

Fig. 5

resulting in an unpredictable outcome has been investigated. Jumps from resonance can also occur in typical non-linear hardening systems. This can be seen in Figures 6(a) and 6(b) for model B.

6. UNSAFE JUMPS AND CHAOTIC DYNAMICS

At a larger value of forcing, as shown in Figure 7(a), one observes some qualitative changes in the response diagram; in particular the frequency value at which the chaotic oscillation loses its stability, at crisis E, has significantly decreased. As shown in Figure 7(b) (i) complex dynamics including enlargement of the chaotic attractor take place after point E. This is accompanied by an increase in the frequency value at which the saddle-node solution loses its stability at point A. In Figure 6(b) (ii) it is shown that long

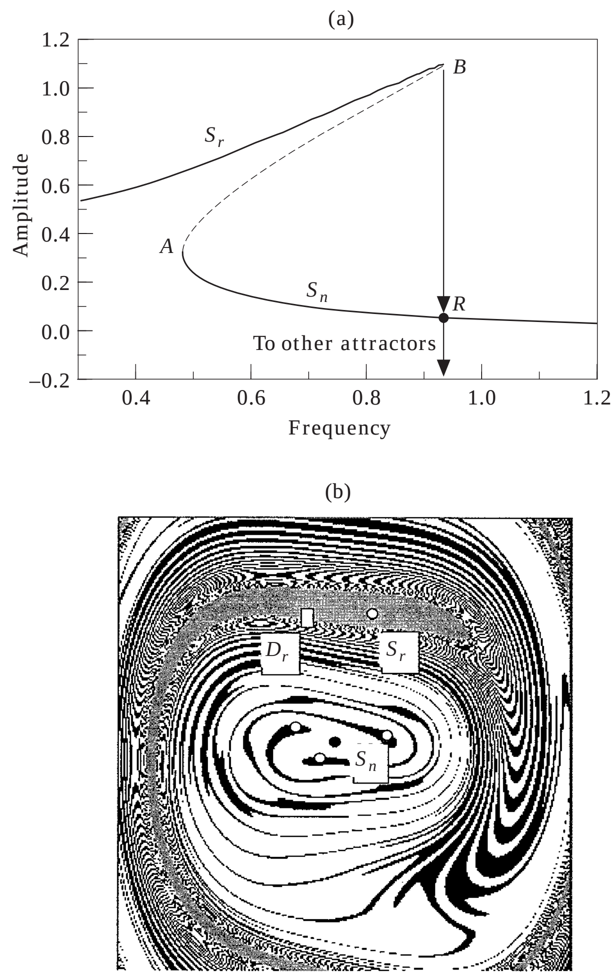


Figure 6. Indeterminate jumps (model B). (a) The steady state resonance response curve at an intermediate forcing level ($F = 0{\cdot}05$). Here the jump to resonance from point B may or may not re-stabilize on to the small amplitude cycle, R, that lies on the non-resonant branch of the response curve. (b) The upper portrait shows the basin structure just prior to an indeterminate subcritical bifurcation in the window $-1{\cdot}5 < x < 1{\cdot}5$, $-1{\cdot}5 < y < 1{\cdot}5$. Here $\omega = 0{\cdot}92$, $F = 0{\cdot}05$. The circles denote the stable mapping points of the fixed points. The colour coding is as follows: light grey represents the basin of attraction of the resonant solution, S_r; white represents the basin of the non-resonant solution, S_r; black is for a coexisting period-3 solution.

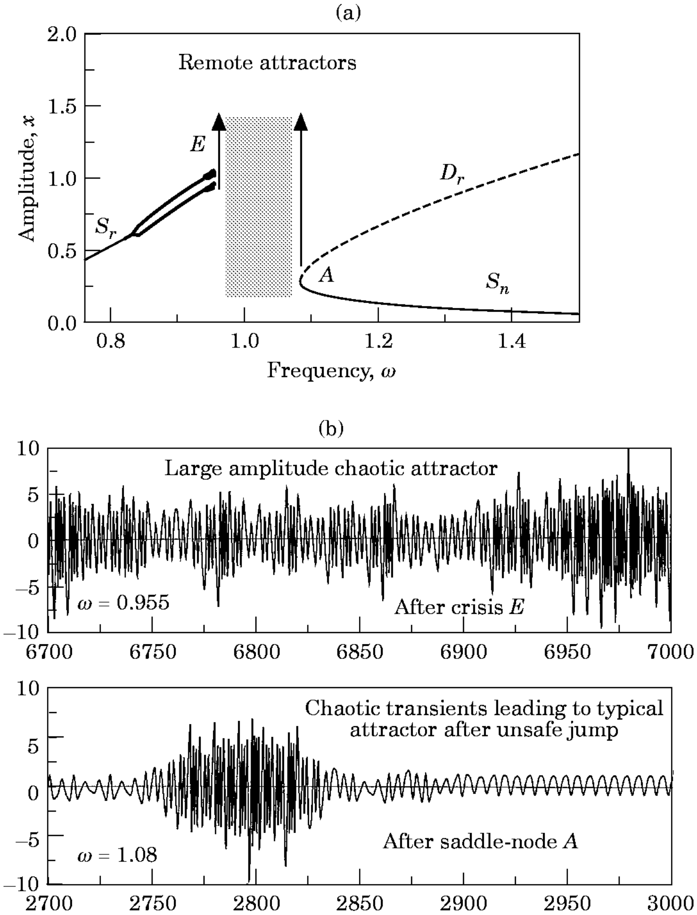


Figure 7. Unsafe jumps. (a) The steady state resonance response curve at a relatively high forcing level ($F = 0{\cdot}99$). Here the jump from point A always results in the system going to a remote attractor. (b) Typical time histories illustrating (i) a large amplitude chaotic attractor after crisis point E, and (ii) long chaotic transients after fold point A.

chaotic transients may occur after fold A before the system typically settles on to one of the remote attractors. This leaves a regime in which there is a fundamental attractor giving an inevitable jump to a remote attractor after the saddle-node bifurcation. In keeping with the ideas presented in earlier papers on softening systems, this is a purely *unsafe* jump.

7. CONCLUSIONS

The non-linear resonance behaviour of typical hardening systems has been outlined. For small forcing levels the system with a linear component (model A) behaves like a linear system, with resonance occurring when the forcing frequency is approximately equal to the linearized natural frequency. As the forcing amplitude is increased the steady state response peaks towards higher frequencies with eventually the well known jump phenomenon taking place. At higher but still moderate forcing levels, it has been shown how unpredictable jumps from indeterminate bifurcations arise as quite typical events in the response of hardening systems. If, at the saddle-node bifurcation, the critical point finds itself on a highly intertwined basin boundary one cannot predict to which of the remote attractors the system will jump since the response is extremely sensitive to how the bifurcation is realized. Examples of unpredictable jumps to and from resonance were presented. If one were to carry out a steady state analysis, neglecting the organization of coexisting basins of attraction prior to the bifurcation, one may seriously overestimate the predictability of the system at the point of instability.

REFERENCES

1. Y. UEDA 1979 *Journal of Statistical Physics* **20**, 181–196. Randomly transitional phenomena in the system governed by Duffing's Equation.
2. T. FANG and E. H. DOWELL 1987 *International Journal of Nonlinear Mechanics* **22**, 401–425. Numerical simulations of periodic and chaotic responses in a stable Duffing system.
3. W. SZEMPLINSKA-STUPNICKA and P. NIEZODZKI 1990 *Journal of Sound and Vibration* **141**, 181–192. The approximate approach to chaos phenomena in oscillators having a single equilibrium position.
4. G. REGA, F. BENEDETTI and A. SALVATORI 1991 *Chaos*, *Solitons & Fractals* **1**, 39–54. Periodic and chaotic motions of an unsymmetrical oscillator in nonlinear structural dynamics.
5. Y. H. KAO, C. S. WANG and T. H. YANG 1992 *Journal of Sound and Vibration* **159**, 13–21. Influences of harmonic coupling on bifurcations in Duffing oscillators with bounded potential well.
6. M. S. SOLIMAN 1993 *Journal of Applied Mechanics* **60**, 669–676. Jumps to resonance: long chaotic transients, unpredictable outcome and the probability of restabilization.
7. J. M. T. THOMPSON & M. S. SOLIMAN 1991 *Proceedings of the Royal Society of London* **A432**, 101–111. Indeterminate jumps to resonance from a tangled saddle-node bifurcation.
8. M. S. SOLIMAN and J. M. T. THOMPSON 1991 *Bifurcation and Chaos* **1**, 107–119. Basin organisation prior to a tangled saddle-node bifurcation.
9. H. B. STEWART and Y. UEDA 1991 *Proceedings of the Royal Society of London* **A432**, 113–123. Catastrophes with indeterminate outcome.
10. M. S. SOLIMAN and J. M. T. THOMPSON 1996 *Proceedings of the Royal Society of London* **A452**, 487–494. Indeterminate bifurcational in hardening systems
11. J. GUCKENHEIMER and P. J. HOLMES 1983 *Nonlinear Oscillations*, *Dynamical Systems*, *and Bifurcations of Vector Fields*. New York: Springer-Verlag.
12. M. S. SOLIMAN 1995 *Proceedings of the Royal Society of London* **A451**, 471–485. Dynamic bifurcations in non-stationary systems: transitions with an unpredictable outcome.
13 M. D. TODD and L. N. VIRGIN 1997 *International Journal of Bifurication and Chaos* [to appear]. An experimental verification of basin metamophoses in a nonlinear mechanical system.
14. L. N. VIRGIN On the harmonic response of an oscillator with unsymmetric restoring force. 1988 *Journal of Sound and Vibration* **126,** 157–165
15. G. REGA, A. SALVATORI and F. BENEDETTI 1992 *Proceedings of the ASME*, *Nonlinear Vibrations* AMD-144, Book No. G00736. Basin bifurcation and chaotic attractor in an elastic oscillator with quadratic and cubic nonlinearities.